

# **Introduction to MHD** (magnetohydrodynamics)

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## What is MHD?

**Magnetohydrodynamics (MHD)** is the standard continuum model for an electrically conducting **fluid** in the presence of a **magnetic field**.

- applies to liquid metals and plasmas.

- main variables:

$\mathbf{v}(\mathbf{x}, t)$  — **velocity field** [m/s] or [cm/s]

$\rho(\mathbf{x}, t)$  — (mass) **density** [kg/m<sup>3</sup>] or [g/cm<sup>3</sup>]

$p(\mathbf{x}, t)$  — (fluid) **pressure** [Pa] or [Ba]

$\mathbf{B}(\mathbf{x}, t)$  — **magnetic field\*** [T] or [G]

\*technically “magnetic flux density”

- key assumptions:

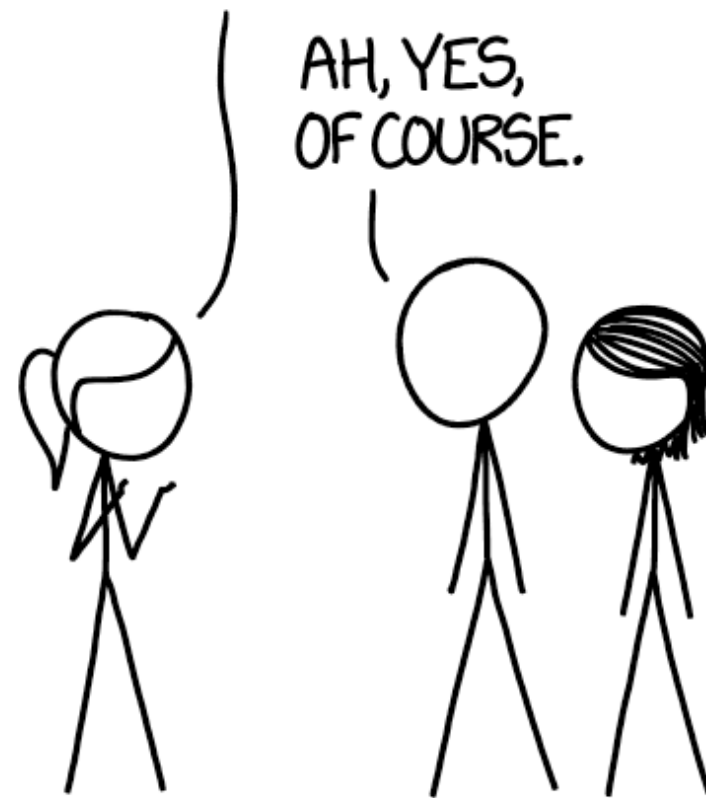
- sufficiently collisional [for fluid model to make sense]

- single fluid [neglect electron mass]

- speeds are sub-relativistic [neglect displacement current...]

**two-way interaction between magnetic field and fluid motions**

THE SUN'S ATMOSPHERE IS A  
SUPERHOT PLASMA GOVERNED BY  
MAGNETOHYDRODYNAMIC FORCES...



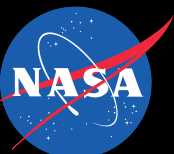
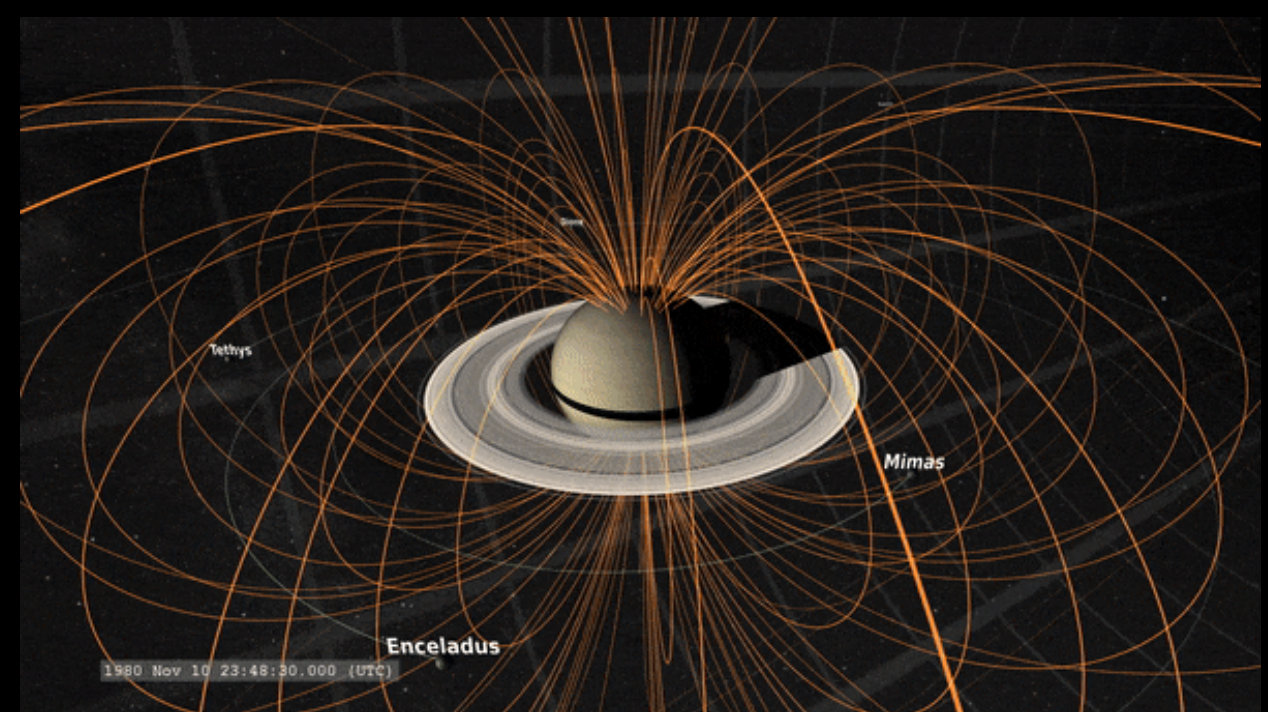
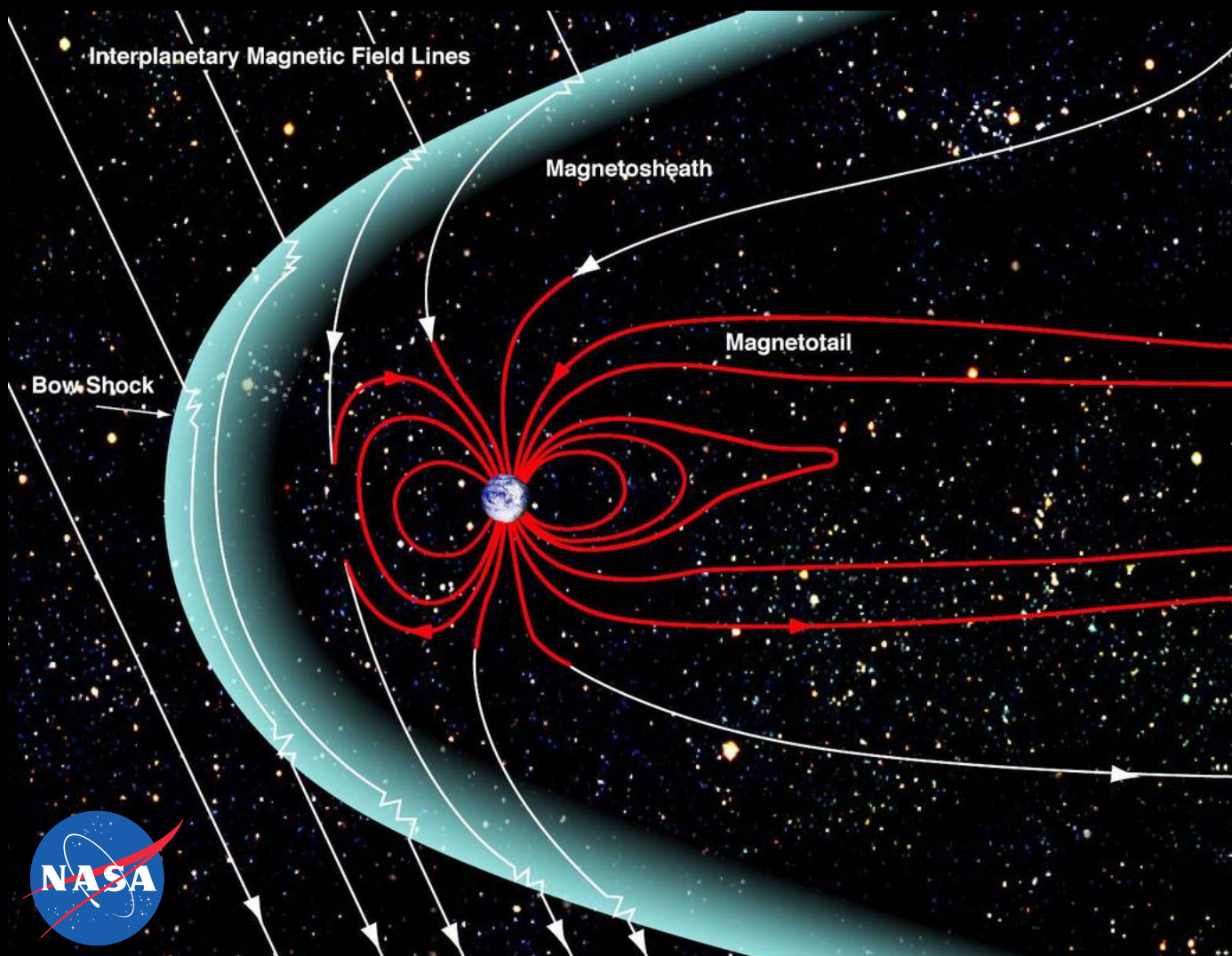
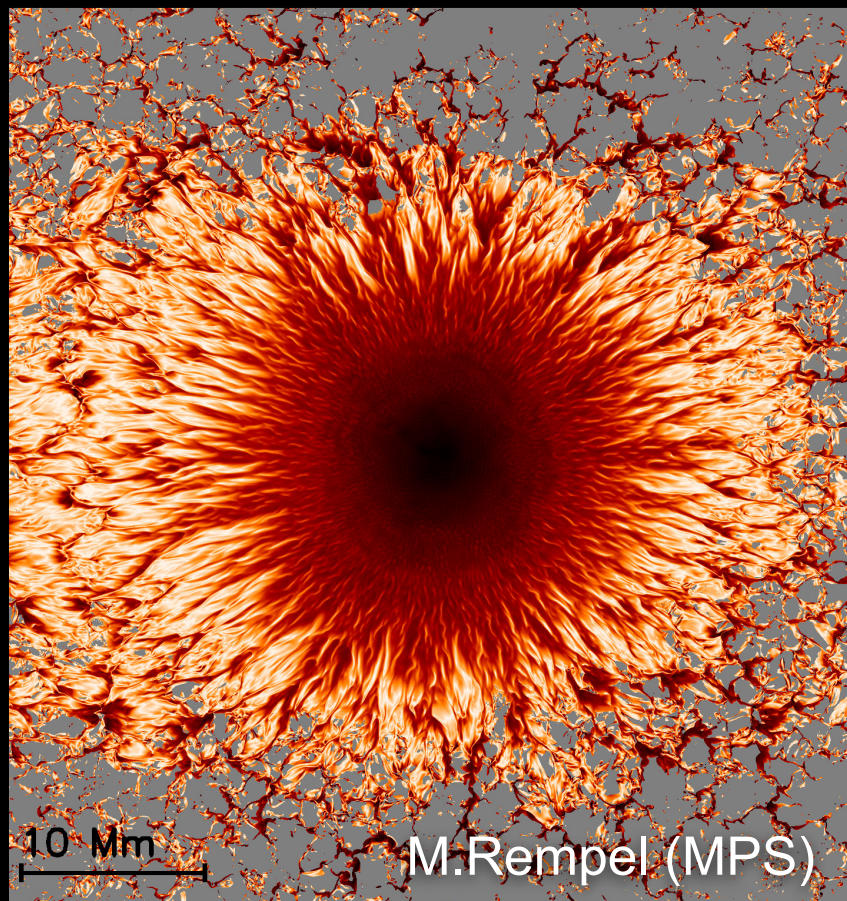
AH, YES,  
OF COURSE.

WHENEVER I HEAR THE WORD  
"MAGNETOHYDRODYNAMIC" MY BRAIN  
JUST REPLACES IT WITH "MAGIC."

<https://xkcd.com/1851/>

"Magnetohydrodynamics combines the intuitive nature of Maxwell's equations with the easy solvability of the Navier-Stokes equations."

# Applications



**ELECTROMAGNETIC EQUATIONS**

**FLUID EQUATIONS**

**EQUILIBRIA**

# Electromagnetic equations

## 1. Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday's Law})$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\text{Ampère's Law})$$



neglect the displacement current [valid for **sub-relativistic** motions]

## 2. Ohm's law:

$$\mathbf{j} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$



electric field in the frame of the moving fluid

conductivity – assume constant but really depends on temperature

Eliminate  $\mathbf{E}$ :

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{v} \times \mathbf{B} - \frac{\mathbf{j}}{\sigma} \right) = \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{\mu_0} \nabla \times \left( \frac{\nabla \times \mathbf{B}}{\sigma} \right)$$



$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \quad \text{– induction equation}$$

magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma}$$

## Implications of the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

If  $\mathbf{v} = \mathbf{0}$ , we have **diffusive decay** of the magnetic field:

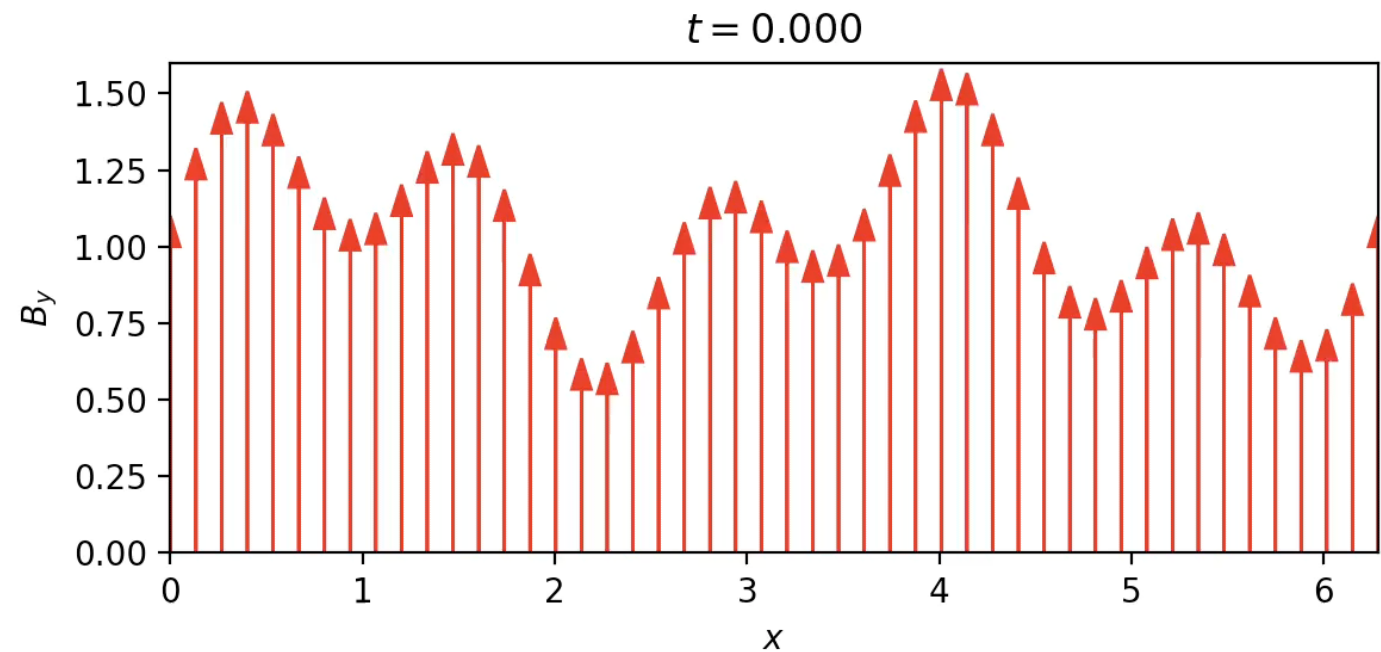
$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$$

- smooths out gradients

e.g.  $\mathbf{B} = \left( B_0 + \sum_k B_k \sin(kx) e^{-\eta k^2 t} \right) \mathbf{e}_y$

- decay time for each “mode” is

$$\tau_\eta \approx \frac{1}{k^2 \eta}$$



To measure when diffusion is important, compare the size of the terms:

typical values

$$\begin{aligned} |\mathbf{v}| &\sim v_0 \\ |\mathbf{B}| &\sim B_0 \end{aligned} \quad \Rightarrow \quad \frac{|\nabla \times (\mathbf{v} \times \mathbf{B})|}{|\eta \nabla^2 \mathbf{B}|} \sim \frac{v_0 B_0 / \ell_0}{\eta B_0 / \ell_0^2} = \frac{v_0 \ell_0}{\eta}$$

magnetic Reynolds' number

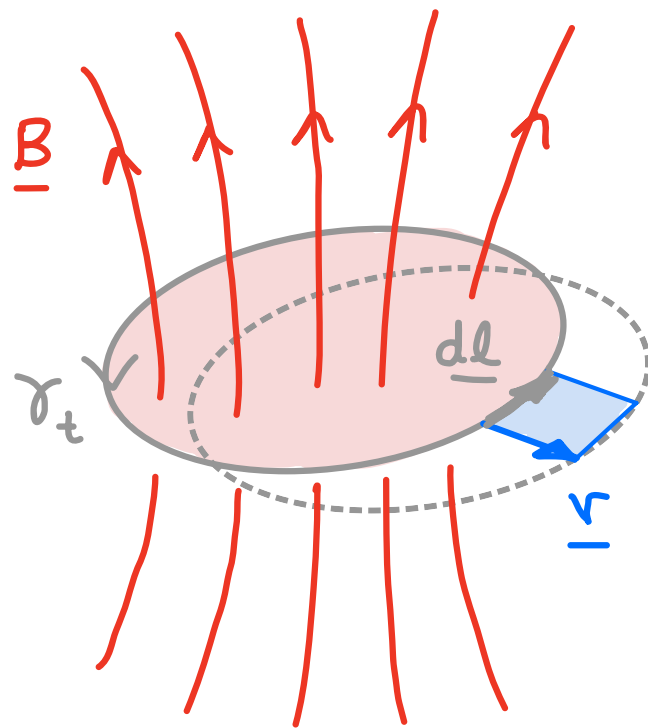
- for Sun, diffusion matters only when  $\mathbf{B}$  varies on small scales [\[prev lecture...\]](#)

## Implications of the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

If  $\eta = 0$ , we have **ideal MHD**:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$



flux through a material curve (moving with fluid)

$$\begin{aligned} \frac{d}{dt} \Phi_{\gamma_t} &= \frac{d}{dt} \int_{S_t} \mathbf{B} \cdot d\mathbf{S} \\ &= \int_{S_t} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_{\gamma_t} \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) \\ &= \int_{S_t} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint_{\gamma_t} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \\ &= \int_{S_t} \left( \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right) \cdot d\mathbf{S} \\ &= 0 \end{aligned}$$

– known as **Alfvén's (frozen flux) Theorem**

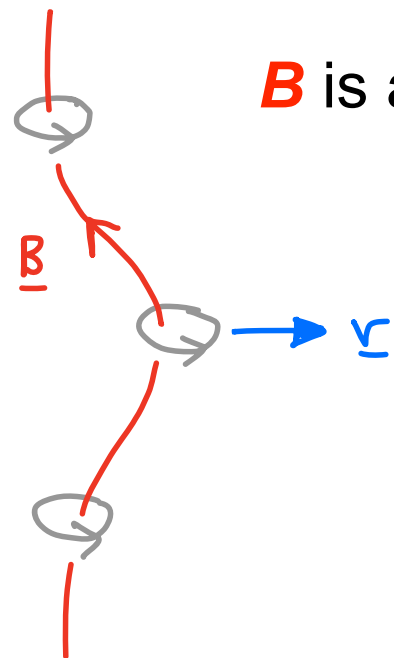
## Hannes Alfvén (1908–1995)

**Nobel Prize for Physics, 1970** – "for fundamental work and discoveries in magnetohydro-dynamics with fruitful applications in different parts of plasma physics"



# Ideal MHD solutions

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

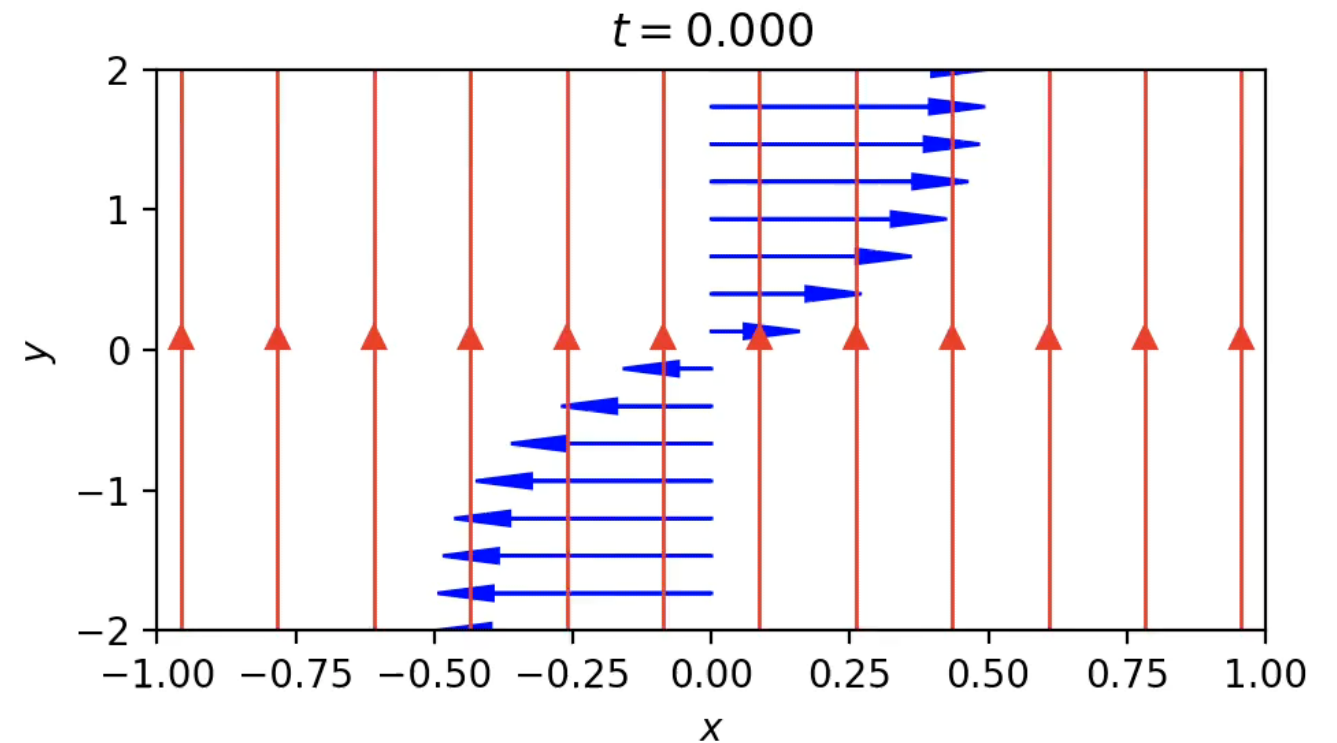


$\mathbf{B}$  is a “material vector”

e.g. (1)

$$\mathbf{v} = u(y)\mathbf{e}_x$$

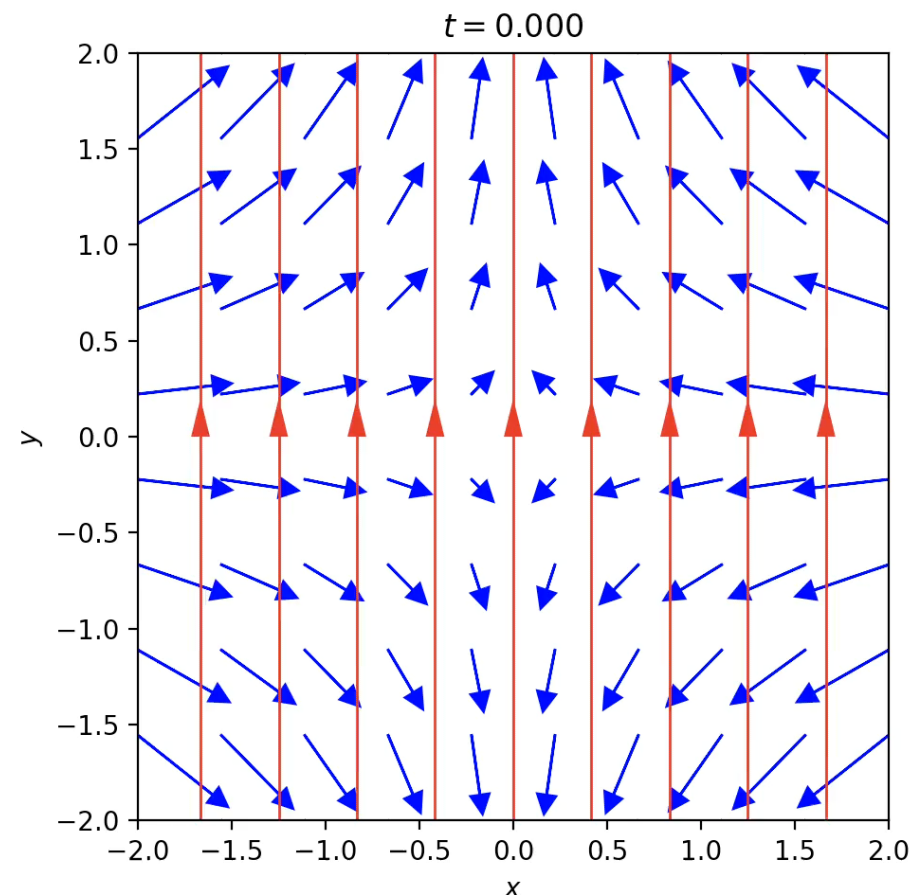
$$\mathbf{B} = B_0 \left[ u'(y)t\mathbf{e}_x + \mathbf{e}_y \right]$$



e.g. (2)

$$\mathbf{v} = v_0(-x\mathbf{e}_x + y\mathbf{e}_y)$$

$$\mathbf{B} = b(xe^{v_0 t})e^{v_0 t}\mathbf{e}_y$$

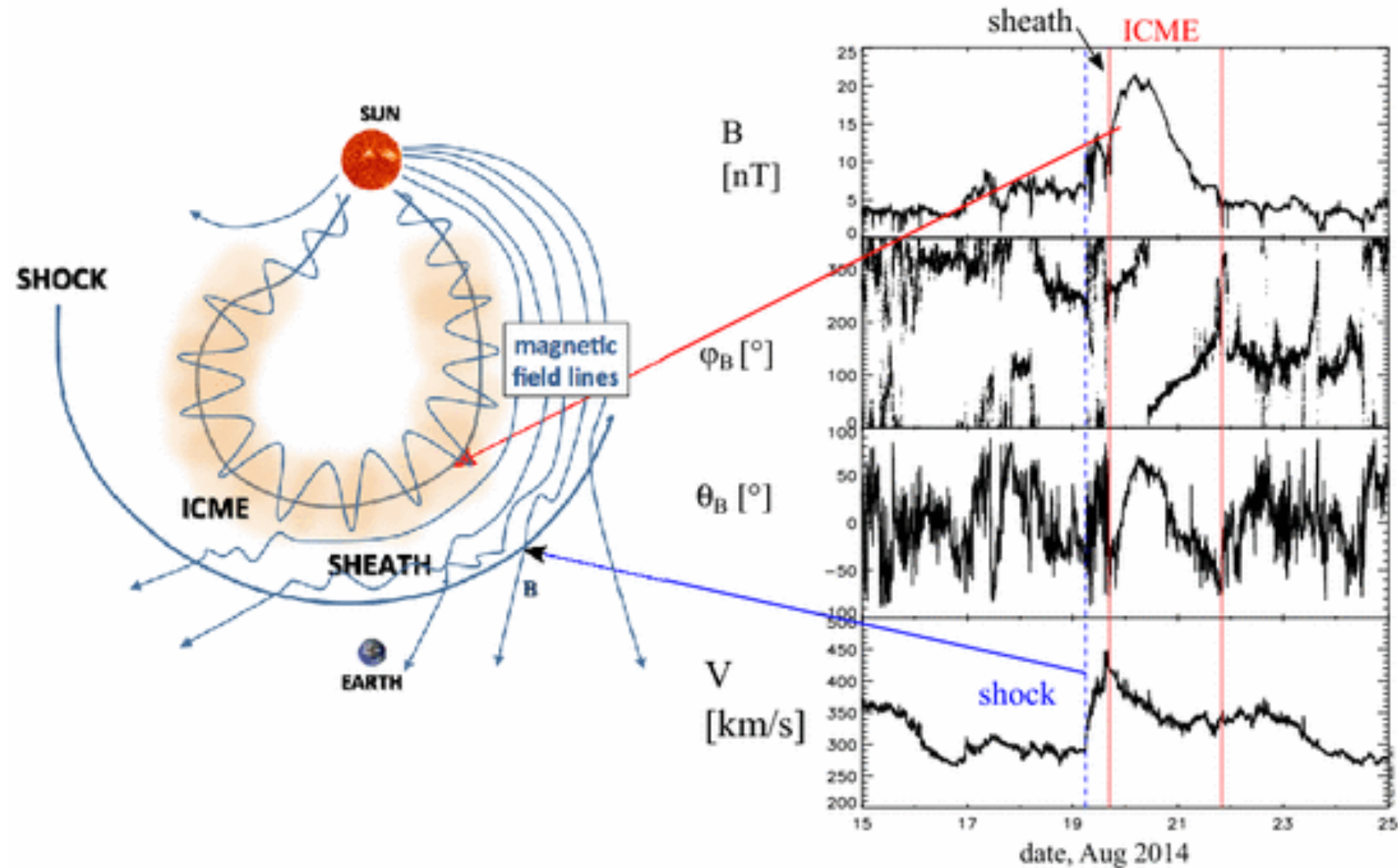


– exponential amplification!

[e.g. solar dynamo –  
Elsworth lecture...]

## Example – magnetic clouds

- twisted “magnetic flux ropes” expelled from the Sun.



Kilpua et al., *Living Rev Solar Phys* (2017)

[see Barnard lecture...]

# Fluid equations

## 3. Momentum equation [usual Navier-Stokes with extra **Lorentz force** term]:

$$\underbrace{\rho \frac{D\mathbf{v}}{Dt}}_{\text{mass [per unit volume] } \times \text{ acceleration}} = -\underbrace{\nabla p + \rho \mathbf{g} + \mathbf{F}_{\text{viscous}} + \mathbf{j} \times \mathbf{B}}_{\text{force [per unit volume]}}$$

mass [per unit volume]  
× acceleration

force [per unit volume]

– material derivative:  $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$

cf. force on a charged particle  $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$   
– neglect Coulomb force for sub-relativistic speeds

## 4. Continuity equation [mass conservation]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \iff \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v} \iff \frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{v}$$

## 5. Energy equation [of choice]:

– e.g. **isothermal**

$$p = c_s^2 \rho$$

constant sound speed

polytropic index  $\gamma = \frac{5}{3}$  (hydrogen)

– e.g. **adiabatic ideal gas**

$$\rho \frac{D\varepsilon}{Dt} = -p \nabla \cdot \mathbf{v}$$

for **internal energy**

$$\varepsilon = \frac{p}{(\gamma - 1)\rho}$$

– e.g. non-adiabatic ideal gas...

## Example – Parker solar wind

Assume Sun's corona is spherically symmetric, isothermal with  $\mathbf{B} \equiv \mathbf{0}$  and  $\mathbf{v} = v(r)\mathbf{e}_r$ .

The (M)HD equations reduce to

$$3. \quad \rho v v' = -p' - \frac{GM_{\odot}\rho}{r^2}$$

$$4. \quad r^2 \rho v = \text{const.}$$

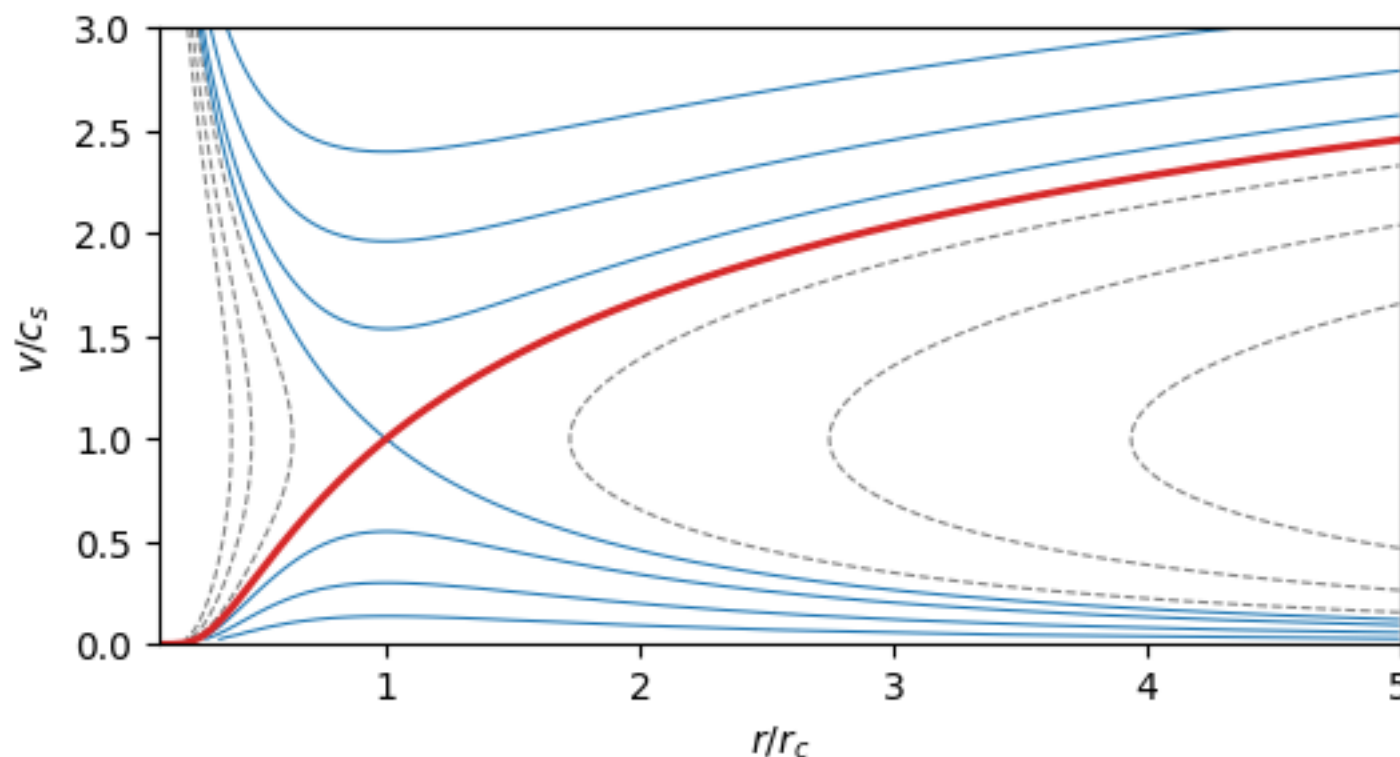
$$5. \quad p = c_s^2 \rho$$

$$\Rightarrow \left( v - \frac{c_s^2}{v} \right) v' = \frac{2c_s^2}{r} - \frac{GM_{\odot}}{r^2}$$

Solving this ODE gives the (implicit) solution

$$\left( \frac{v}{c_s} \right)^2 - \log \left( \frac{v}{c_s} \right)^2 = 4 \log \frac{r}{r_c} + \frac{4r_c}{r} + C$$

“critical radius”  $r_c \equiv \frac{GM_{\odot}}{2c_s^2}$



– only physical solution [subsonic at Sun and no pressure at large distances] is  $C = -3$ .

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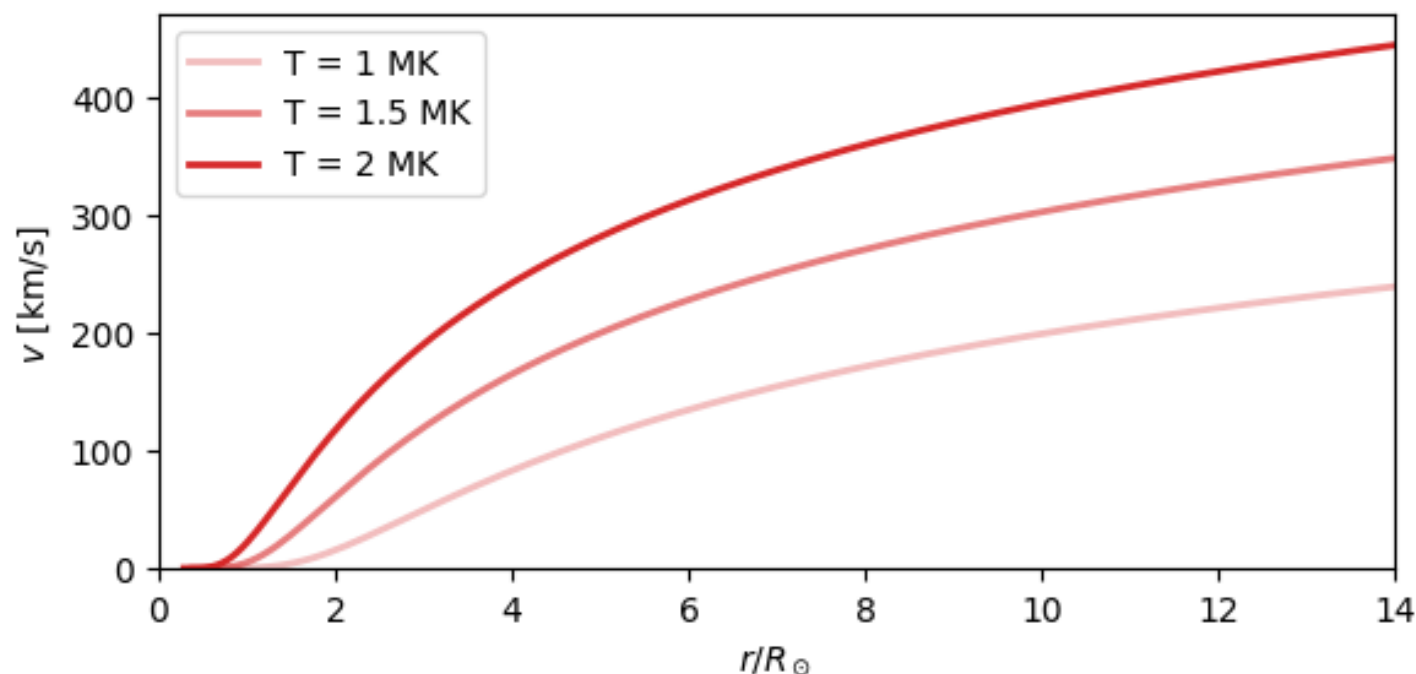
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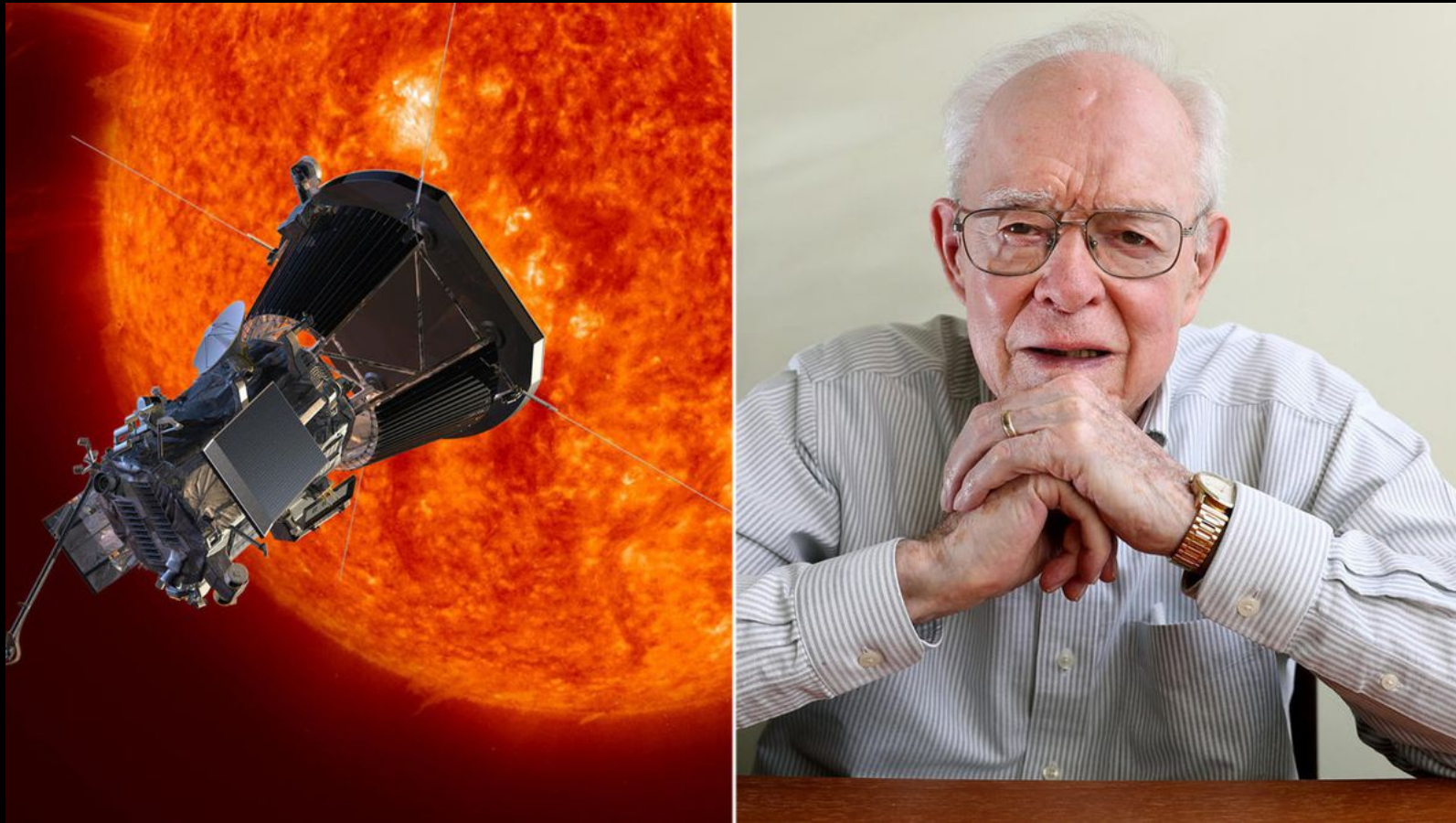
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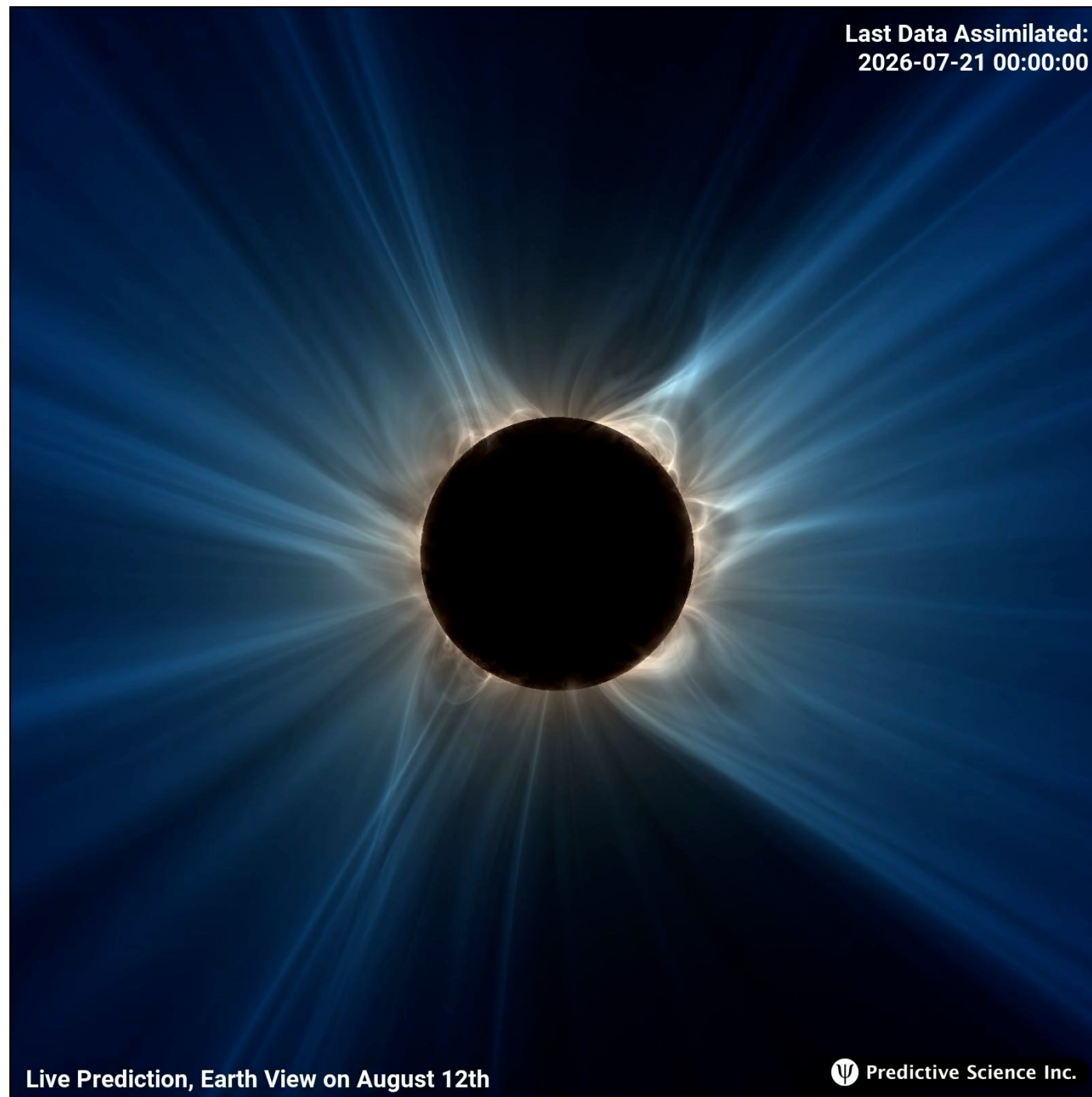
[see Barnard lecture...]

## Eugene Parker (1927–2022)



## Example – eclipse prediction

<https://eclipse.predsci.com/>

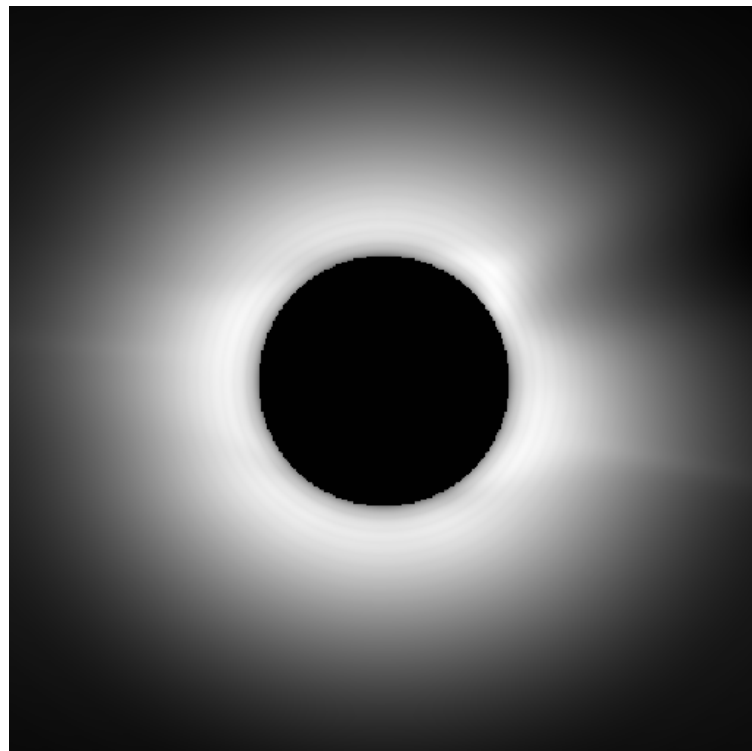


cf. Downs et al., *Science* (2025)

# Thermodynamics matter!

from <https://www.predsci.com/mhdweb/>

“polytropic”

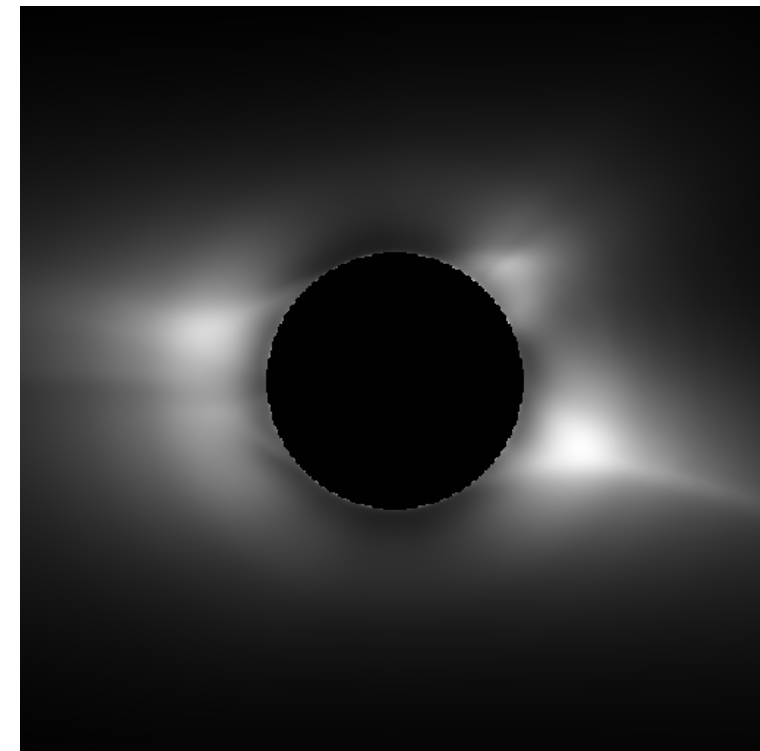


$$\rho \frac{D\varepsilon}{Dt} = -p \nabla \cdot \mathbf{v}$$

(adiabatic)

vs.

“thermodynamic”



$$\rho \frac{D\varepsilon}{Dt} = -p \nabla \cdot \mathbf{v} + \underbrace{\nabla \cdot (\kappa \nabla T)}_{\text{heat conduction}} - \underbrace{\rho^2 Q(T)}_{\text{radiation}} + \underbrace{\frac{|\mathbf{j}|^2}{\sigma}}_{\text{ohmic heating}} + \dots$$

temperature  $T = \frac{m\varepsilon}{k_B(\gamma - 1)}$

## Magnetic equilibria

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g} + \mathbf{F}_{\text{viscous}} + \mathbf{j} \times \mathbf{B}$$

When is neglecting the left-hand side valid?

Typical values:  $|\rho| \sim \rho_0$      $|\mathbf{v}| \sim v_0$      $t \sim \frac{\ell_0}{v_0}$      $|\mathbf{B}| \sim B_0$      $|p| \sim p_0$

$$\left| \rho \frac{D\mathbf{v}}{Dt} \right| \ll |\nabla p| \quad \Longleftrightarrow \quad \frac{\rho_0 v_0^2}{\ell_0} \ll \frac{p_0}{\ell_0} \quad \Longleftrightarrow \quad v_0 \ll \underbrace{\sqrt{\frac{p_0}{\rho_0}}}_{\text{sound speed}}$$

$$\left| \rho \frac{D\mathbf{v}}{Dt} \right| \ll |\mathbf{j} \times \mathbf{B}| \quad \Longleftrightarrow \quad \frac{\rho_0 v_0^2}{\ell_0} \ll \frac{B_0^2}{\mu_0 \ell_0} \quad \Longleftrightarrow \quad v_0 \ll \underbrace{\frac{B_0}{\sqrt{\mu_0 \rho_0}}}_{\text{Alfvén speed}}$$

– conversely, must be rough speed of  $\mathbf{B}$ -driven motions...

– magnetohydrostatic equilibrium

$$\mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g} = 0$$

## Force-free equilibria

$$\mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g} = \mathbf{0}$$

– can neglect gravity if

$$|\rho \mathbf{g}| \ll |\nabla p| \iff \rho_0 g \ll \frac{p}{\ell_0} \iff \ell_0 \ll \underbrace{\frac{p_0}{\rho_0 g}}_{\text{pressure scale height}}$$

– can neglect pressure gradients if

$$|\nabla p| \ll |\mathbf{j} \times \mathbf{B}| \iff \frac{p_0}{\ell_0} \ll \frac{B_0^2}{\mu_0 \ell_0} \iff \underbrace{\frac{2\mu_0 p_0}{B_0^2}}_{\text{plasma beta}} \ll 1$$

– force-free equilibrium

$$\mathbf{j} \times \mathbf{B} = \mathbf{0}$$

$$\iff \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

$$0 = \nabla \cdot (\alpha \mathbf{B}) \iff \mathbf{B} \cdot \nabla \alpha = 0 \quad \text{so } \alpha \text{ is constant along magnetic field lines}$$

1.  $\nabla \alpha \neq 0$

nonlinear force-free field

2.  $\alpha(\mathbf{x}) = \alpha_0$

linear force-free field

3.  $\alpha \equiv 0$

potential field  
( $\mathbf{j} = \mathbf{0}$ )

## Example – potential fields

$$j = 0$$

$$\nabla \times \mathbf{B} = \mathbf{0} \implies \mathbf{B} = \nabla \psi$$

$$\nabla \cdot \mathbf{B} = 0 \implies$$

$$\nabla^2 \psi = 0$$

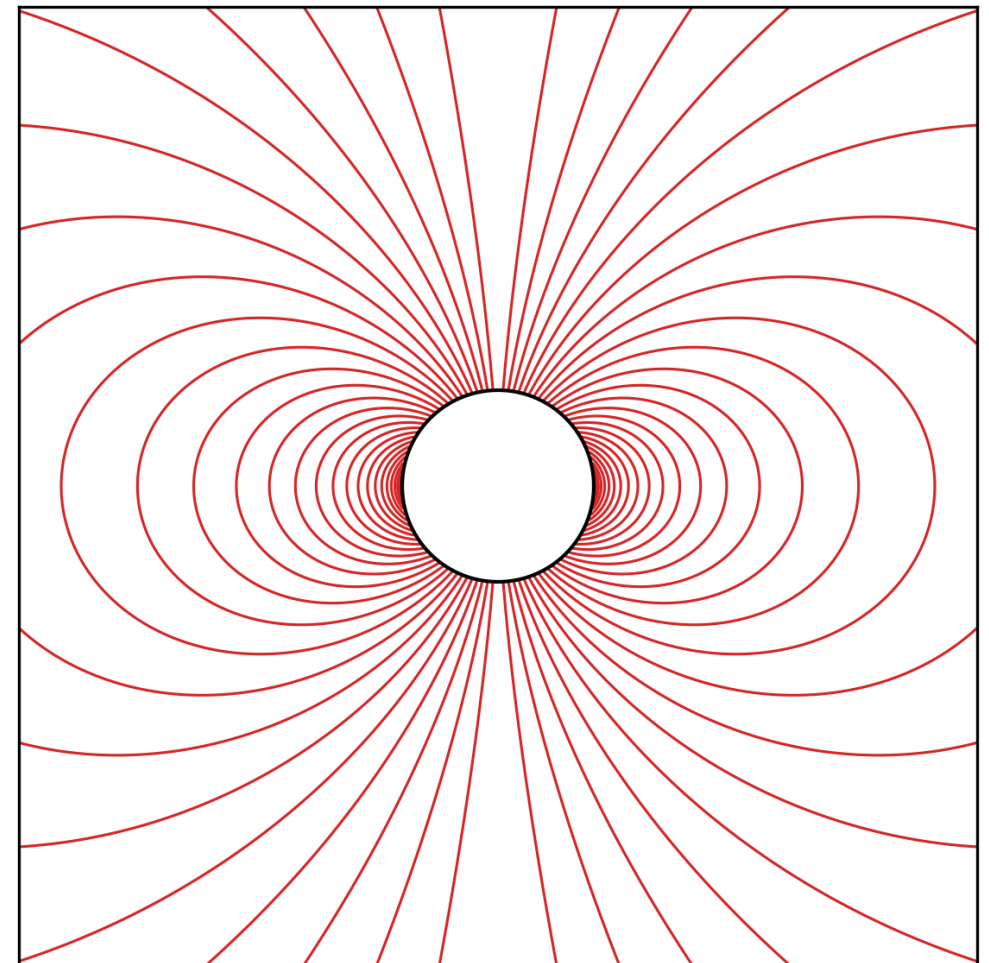
← Laplace equation

– inner boundary:  $\left. \frac{\partial \psi}{\partial r} \right|_{r=R_\odot} = \left. B_r \right|_{r=R_\odot} = b_\odot(\theta, \phi)$  [Neumann boundary condition]

– outer boundary: “infinite space” [decaying modes only]

e.g. dipole  $b_\odot(\theta, \phi) = B_0 \cos \theta$

$$\mathbf{B} = B_0 \left[ 2 \left( \frac{R_\odot}{r} \right)^3 \cos \theta \mathbf{e}_r + \left( \frac{R_\odot}{r} \right)^3 \sin \theta \mathbf{e}_\theta \right]$$



## Example – potential fields

$$j = 0$$

$$\nabla \times \mathbf{B} = \mathbf{0} \implies \mathbf{B} = \nabla \psi$$

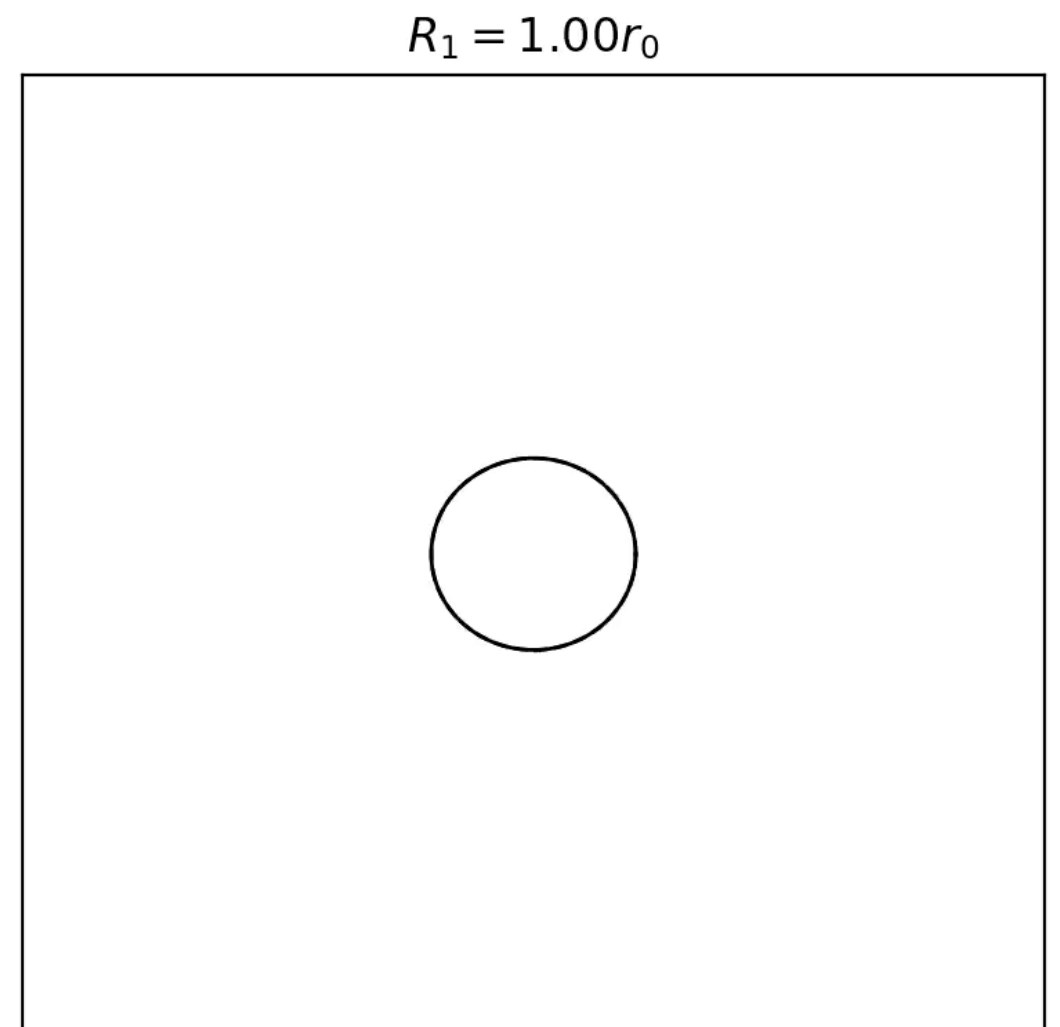
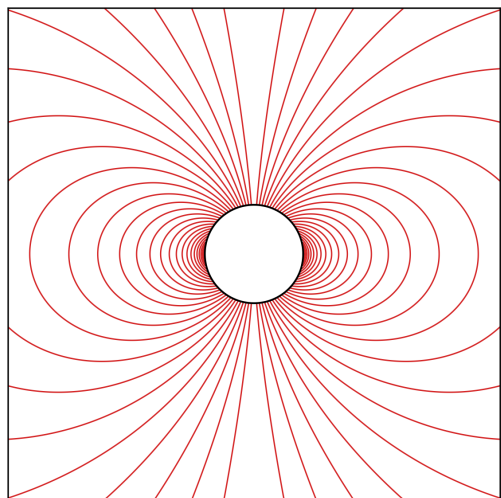
$$\nabla \cdot \mathbf{B} = 0 \implies \nabla^2 \psi = 0 \quad \leftarrow \text{Laplace equation}$$

– inner boundary:  $\left. \frac{\partial \psi}{\partial r} \right|_{r=R_\odot} = \left. B_r \right|_{r=R_\odot} = b_\odot(\theta, \phi)$  [Neumann boundary condition]

– outer boundary:  $\left. \mathbf{e}_r \times \nabla \psi \right|_{r=R_1} = \mathbf{0}$  [PFSS/potential field source surface model]

e.g. dipole  $b_\odot(\theta, \phi) = B_0 \cos \theta$

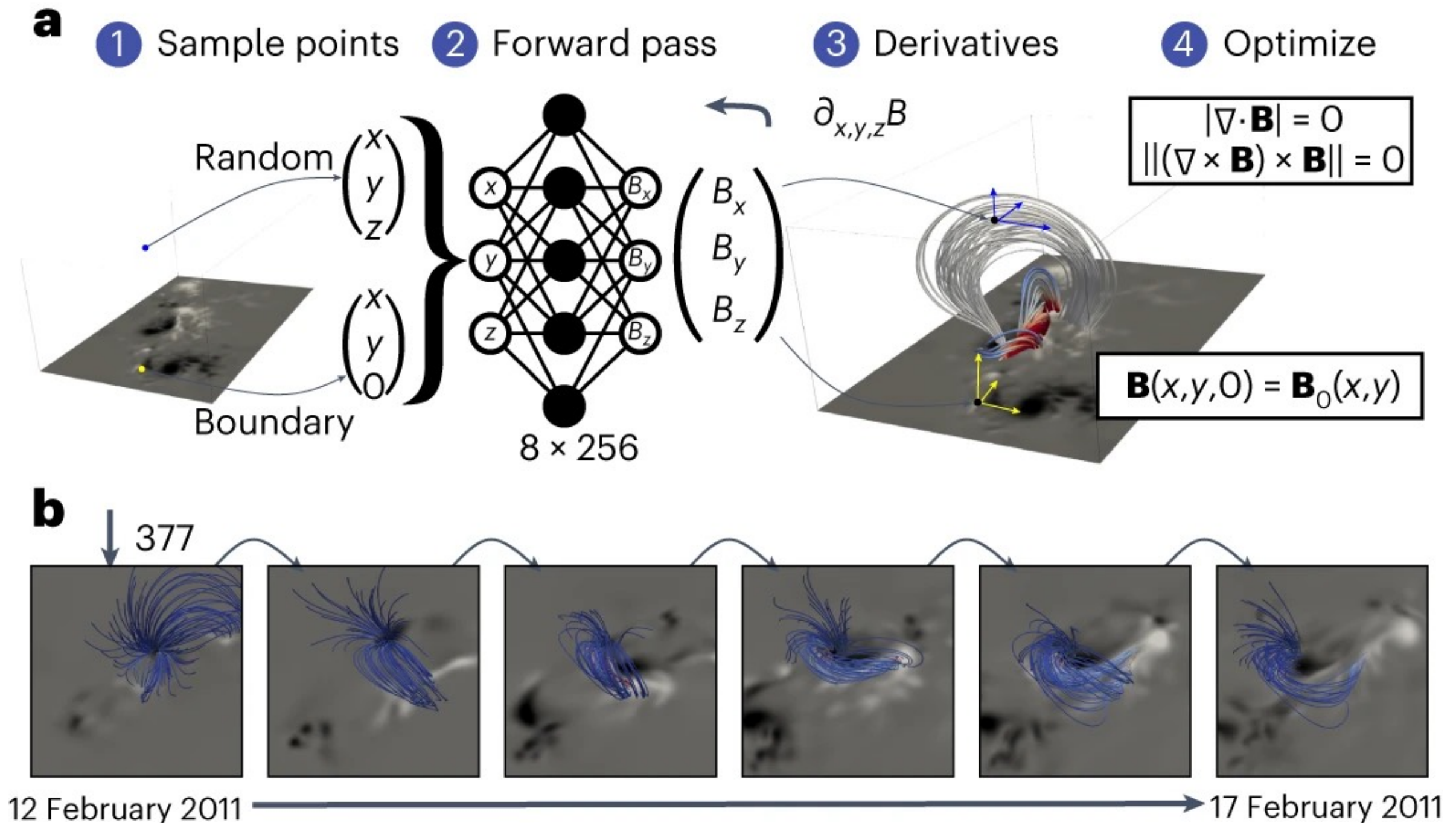
$$\mathbf{B} = B_0 \left[ \frac{R_\odot^3 (r^3 + 2R_1^3)}{r^3 (R_\odot^3 + 2R_1^3)} \cos \theta \mathbf{e}_r + \frac{R_\odot^3 (R_1^3 - r^3)}{r^3 (R_\odot^3 + 2R_1^3)} \sin \theta \mathbf{e}_\theta \right]$$



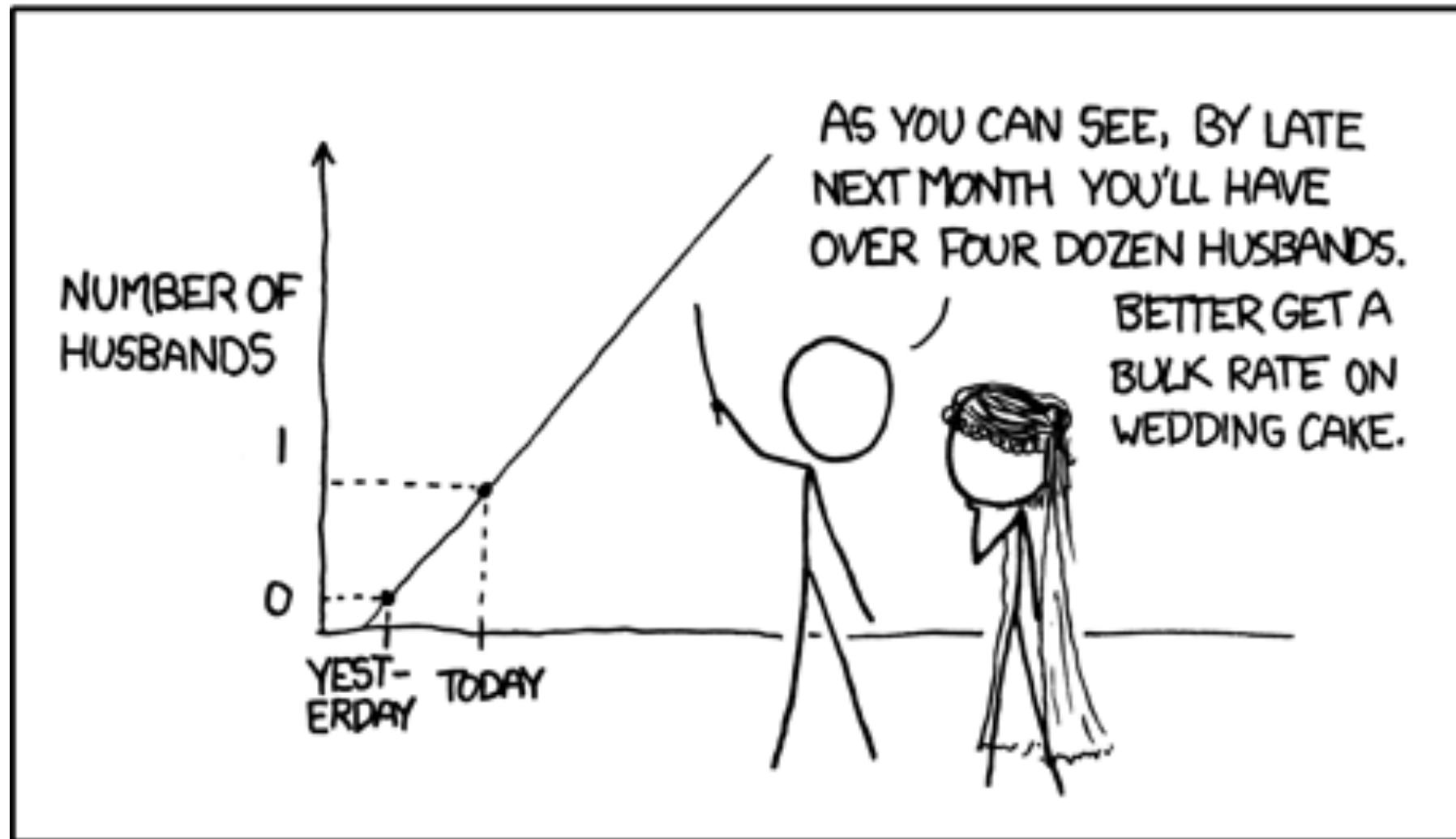
## Example – NLFFF models for active regions

$$\mathbf{j} \times \mathbf{B} = 0$$

– challenge: observations don't give a unique solution



## MY HOBBY: EXTRAPOLATING



<https://xkcd.com/1851/>

## Summary

MHD equations

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g} + \mathbf{F}_{\text{viscous}} + \mathbf{j} \times \mathbf{B}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \rho \frac{D\epsilon}{Dt} = -p \nabla \cdot \mathbf{v} + \dots$$

### Further reading:

- My lecture notes from the Durham 4th year course:  
[https://www.maths.dur.ac.uk/users/anthony.yeates/NOTES/mhd\\_notes/\\_book/](https://www.maths.dur.ac.uk/users/anthony.yeates/NOTES/mhd_notes/_book/)
- *Essential Magnetohydrodynamics for Astrophysics*, H. Spruit:  
<https://arxiv.org/abs/1301.5572>
- *Introduction to Modern Magnetohydrodynamics*, S. Galtier, Cambridge.
- *Magnetohydrodynamics of Laboratory and Astrophysical Plasmas*, H. Goedbloed, R. Keppens & S. Poedts, Cambridge.
- *Physics of Space Plasma Activity*, K. Schindler, Cambridge.
- *An Introduction to Magnetohydrodynamics*, P.A. Davidson, Cambridge.
- *Magnetohydrodynamics of the Sun*, E.R. Priest, Cambridge.